FINDING THE CHARACTERISTIC EXPONENTS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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PMM Vol.22, No.3, 1958, pp.382-385

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(Received 2 February 1956)

Let us consider a system of differential equations of the type

$$\frac{dx_s}{dt} = \sum_{i=1}^{n} (a_{si} + \mu f_{si}(t, \mu))x_i \qquad (s=1, \ldots, n) \tag{1}$$

Here the a_{si} are constants, the f_{si} are continuous, periodic functions of period ω in t, and analytic functions of the parameter μ in some region $|\mu| < \mu^*$.

The problem on the finding of the characteristic exponents for the system (1) has been considered in Ref. [1-3].

We denote the roots of the equation

$$|a_{si} - \delta_{si}\lambda| = 0 \tag{2}$$

by $\lambda_i (i = 1, \ldots, n)$.

Let us consider the root λ_1. It has been shown that if λ_1 is a simple root, and if the differences

$$\lambda_i - \lambda_1 \qquad (i = 2, \ldots, n)$$

are different from the numbers of the type

$$\pm N\sqrt{-1}\ 2\pi/\omega \qquad (N \text{ is an integer})$$

then the characteristic exponent corresponding to λ_1 will be an analytic function of the parameter μ. One can find it by attempting to satisfy the system (1) with a solution of the form

$$x_s = (x_s^{(0)} + \mu x_s^{(1)} + \cdots) \exp(\lambda_1 + \mu\alpha_1 + \mu^2\alpha_2 + \cdots)t \tag{3}$$

where $\alpha_1, \alpha_2, \ldots$ are unknowns which are to be determined from the condition of periodicity of the unknown periodic functions $x_s^{(1)}, x_s^{(2)}, \ldots$

The case when λ_1 is a multiple root, and when some of the differences $\lambda_i - \lambda_1$ are numbers of the form $\pm N\sqrt{-1}\,2\pi/\omega$ (N is an integer), has not been considered. It is the object of the present note to treat this case.

We assume that the roots, which are either equal to λ_1 or differ from it by numbers of the form $\pm N\sqrt{-1}\,2\pi/\omega$, have prime elementary divisors. Let the number of these roots and λ_1 be equal to $m \leqslant n$. Then the system of equations

$$\frac{dx_s^{(0)}}{dx} = \sum_{i=1}^{n} a_{si}\, x_i^{(0)} - \lambda_1 x_s^{(0)} \qquad (s=1,\ldots,n) \tag{4}$$

has m periodic solutions $\phi_{s1}, \ldots, \phi_{sm}$ of period ω. Let us denote by $\psi_{s1}, \ldots, \psi_{sm}$ these periodic solutions of the system (4) for which

$$\psi_{ij}\,\varphi_{ik} + \cdots + \psi_{nj}\,\varphi_{nk} = \delta_{jk} \qquad (\delta_{ii} = 1,\ \delta_{ij} = 0,\ i \neq j)$$

Theorem I. To the root $\alpha = \alpha^*$ of the equation

$$\Delta(\alpha) \equiv |\, b_{ij} - \delta_{ij}\,\alpha \,| = 0 \qquad (i,\ j = 1, \ldots, m) \tag{5}$$

$$\left(b_{ij} = \frac{1}{\omega}\int_0^{\omega} \sum_{i=1}^{n} \sum_{\sigma=1}^{n} f_{s\sigma}(t,\ 0)\,\varphi_{\sigma j}\,\psi_{si}\,dt\right)$$

there corresponds the analytic (relative to μ) characteristic exponent

$$\lambda_1 + \mu\alpha^* + \mu^2\alpha_2 + \cdots$$

We show first of all the method for finding the characteristic exponent in this case. We are going to attempt to determine the particular solution in the form (3). Substituting the expression given by (3) into the equation (1) and equating coefficients of like powers of μ, we find that the functions $x_s^{(0)}$ satisfy the system of equations (4) while the remaining functions $x_s^{(1)}$, $x_s^{(2)}$, ... satisfy the systems

$$\frac{dx_s^{(1)}}{dt} = \sum_{i=1}^{n} a_{si}\, x_i^{(1)} - \lambda_1 x_s^{(1)} + \sum_{i=1}^{n} f_{si}(t,\ 0)\, x_i^{(0)} - \alpha_1 x_s^{(0)}. \tag{6}$$

. . . .

From (4) we obtain

$$x_s^{(0)} = \beta_1^{(0)}\varphi_{s1} + \cdots + \beta_m^{(0)}\cdot\varphi_{sm}$$

where the $\beta_i^{(0)}$ are arbitrary constants. In order that the system (6) may also have a periodic solution it is necessary and sufficient that the following conditions be satisfied

$$b_{i1}\,\beta_1^{(0)} + \cdots + b_{im}\beta_m^{(0)} - \alpha_1\beta_i^{(0)} = 0 \qquad (i=1,\ldots,m) \tag{7}$$

These conditions represent a system of linear, homogeneous equations in the $\beta_i^{(0)}$. If this system is to have a nontrivial solution, α_1 has to be one of the roots of equations (5). Let us suppose that $\alpha_1 = \alpha^*$. We denote the algebraic cofactor of the element $b_{ij} - \delta_{ij}\alpha^*$ of the determinant $\Delta(\alpha^*)$ by $\Delta_{ij}(\alpha^*)$. Since $d\Delta(\alpha)/d\alpha \neq 0$ when $\alpha = \alpha^*$, it follows that we may suppose (without loss of generality) that $\Delta_{mm}(\alpha^*) \neq 0$. Solving the system (7) for $\beta_i^{(0)}$, we obtain

$$\beta_i^{(0)} = \frac{\Delta_{mi}(\alpha^*)}{\Delta_{mm}(\alpha^*)}\, c$$

where c is an arbitrary constant. The periodic solution of the system (6) can now be found in the form

$$x_s^{(1)} = \beta_1^{(1)}\varphi_{s1} + \cdots + \beta_m^{(1)}\varphi_{sm} + \Phi_s^{(1)}(t)\, c$$

where $\phi_s(t)$ is a periodic function of t, and the $\beta_i^{(1)}$ are arbitrary constants.

Next, substituting $x_s^{(1)}$ into the equations which determine the function $x_s^{(2)}$, we obtain the conditions for the existence of a periodic solution for these equations in the form

$$b_{i1}\beta_1^{(1)} + \cdots + b_{im}\beta_m^{(1)} - \alpha_2\beta_i^{(0)} + c\, A_i^{(1)} = 0 \qquad (i = 1, \ldots, m) \tag{8}$$

where the $A_i^{(1)}$ are definite constants. This is a system of m linear nonhomogeneous equations in the $m + 1$ unknowns $\beta_1^{(1)}, \ldots, \beta_m^{(1)}, \alpha_2$.

It is easy to establish the equation

$$\begin{vmatrix} b_{11} - \alpha^* \cdots b_{1\, m-1} & \beta_1^{(0)} \\ b_{21} \quad \cdots b_{2\, m-1} & \beta_2^{(0)} \\ \cdots\cdots\cdots & \cdots \\ b_1 \cdots\cdots b_{m\, m-1} & \beta_m^{(0)} \end{vmatrix} = \left[\frac{d\Delta(\alpha)}{d\alpha}\right]_{\alpha=\alpha^*} \qquad c \neq 0$$

The system (8) can, therefore, be solved for $\beta_1^{(1)}, \ldots, \beta_{m-1}^{(1)}$, and α_2 in terms of $\beta^{(1)}$ and c. In doing this it will be found that α_2 does not depend on $\beta_1^{(1)}$ or c:

$$\alpha_2 = \frac{1}{\Delta'(\alpha^*)} \begin{vmatrix} b_{11} - \alpha^* \cdots b_{1\, m-1} & A_1^{(1)} \\ \cdots\cdots\cdots & \cdots \\ b_{m1} \cdots\cdots b_{m\, m-1} & A_m^{(1)} \end{vmatrix}$$

Making use of the fact that the solution (3) of the system (1) is determined up to an arbitrary factor, we set $\beta_m^{(1)} \equiv 0$.

The functions $x_s^{(2)}$, $x_s^{(3)}$ are found in a manner analogous to the one by which $x_s^{(1)}$ was determined.

The constants $\beta_1^{(k)}, \ldots, \beta_m^{(k)}$ and α_{k+1} satisfy a system which differs from the system (8) only by one term. We now find the α_{k+1}. Setting $\beta_m^{(k)} = 0$, we express $\beta_1^{(k)}, \ldots, \beta_{m-1}^{(k)}$ in terms of c, and so on.

According to theorem 1, it follows from the uniqueness of the expansion

$\lambda_1 + a^*\mu + a_2\mu^2 + \ldots$, that we have found a series which converges and represents the characteristic exponent. The series $x_s^{(0)} + \mu x_s^{(1)} + \ldots$ can be divergent. In order that these series might converge it is necessary to make a special selection of the initial values $\beta_m^{(k)}$. Without loss of generality, one may assume that $\phi_{1m}(0) \neq 0$. Then, in order that the series (3) converge, it is sufficient to select the $\beta_m^{(1)}$, $\beta_m^{(2)}$, $\beta_m^{(3)}$, ... so that the series

$$\mu x_1^{(1)}(0) + \mu^2 x_1^{(2)}(0) + \mu^3 x_1^{(3)}(0) + \ldots$$

converge. This can obviously be done.

Proof. We shall look for a periodic solution, of the system of linear equations, of the type

$$\frac{dx_s}{d} = \sum_{i=1}^{n} a_{si} x_i - \lambda_1 x_s + \mu \sum_{i=1}^{n} f_{si}(t, \mu) x_i - \mu \alpha x_s \qquad (s=1,\ldots, n) \tag{9}$$

It is obvious that, from the condition for the existence of a periodic solution of this system, we can find the characteristic exponent $\lambda_1 + \mu a$.

By means of a linear non-singular transformation we can reduce the system (9) to the form

$$\frac{du_i}{dt} = \mu (P_{i1} u_1 + \cdots + p_{im} u_m + p_{i,\, m+1} v_1 + \cdots p_{in} v_l) - \mu \alpha u_i$$

$$\frac{dv_j}{dt} = \sum_{\sigma=1}^{n} c_{j\sigma} v_\sigma + \mu (q_{j1} u_1 + \cdots + q_{jm} u_m + q_{j\ m+1} v_1 + \cdots + q_{in} v_l) - \mu \alpha v_j \tag{10}$$

$$(i = 1, \ldots, m;\ j = 1, \ldots, l;\ m + l = n,\ C_{j6} = \text{const})$$

where $u_i = \psi_{i1} x_1 + \ldots + \psi_{ni} x_h$ are functions of p and q of the same type as $f_{\sigma i}$.

Let us consider the auxiliary system

$$\frac{du_i}{dt} = \mu (p_{i1} u_1 + \cdots + p_{im} u_m + p_{i\ m+1} v_1 + \cdots + p_{in} v_l) - \mu \alpha u_i + W_i$$

$$\frac{dv_j}{dt} = \sum_{\sigma=1}^{l} c_{j\sigma} v_\sigma + \mu (q_{j1} u_1 + \cdots + q_{jm} u_m + q_{j\ m+1} v_1 + \ldots + q_{jn} v_l) - \mu \alpha v_i \tag{11}$$

$$(i = 1, \ldots, m;\ j = 1, \ldots, l;\ m + l = n)$$

where the W_i are constants. In Ref. [3] it is shown that the system (11) admits a periodic solution, analytic in μ and of the form

$$u_i = u_i^{(0)} + \mu u_i^{(1)} + \cdots, \quad v_j = v_j^{(1)} \mu + v_j^{(2)} \mu^2 + \cdots$$

The constants W_i are uniquely determined from the condition of the existence of a periodic solution for the system (11) of the form

$$W_i = W_i^{(1)}\mu + W_i^{(2)}\mu^2 + \cdots$$

The first m functions u have arbitrary initial values $u_i(0) = \beta_i$ (β_i are constants). The initial values for the functions v_j are determined by the periodicity of the v_j.

The series thus constructed will converge in a neighborhood of $\mu = 0$.

Let us suppose that we have found a periodic solution of the system (11) and have determined the corresponding W_i. These W_i will be linear homogeneous functions of $\beta_1, \ldots, \beta_m$ and analytic functions of the parameters μ and α.

In order that the system (10) can have a periodic solution it is necessary and sufficient that the following system of linear homogeneous equations in β_i be satisfied:

$$W_i(\beta_1, \ldots, \beta_m, \mu, \alpha\mu) = 0 \quad (i = 1, \ldots, m) \tag{12}$$

This will be the case if

$$\frac{\partial(W_1, \ldots, W_m)}{\partial(\beta_1, \ldots, \beta_m)} = 0 \tag{13}$$

The condition (13) represents an equation of the form $F(\mu, \alpha\mu) = 0$.

From this equation we find m solutions $\alpha_i(\mu)$ which give us m characteristic exponents $\lambda_1 + \mu\alpha_i(\mu)$ $(i = 1, \ldots, m)$.

Carrying out the suggested operations we obtain

$$\begin{gathered} W_i = -\frac{\mu}{\omega}\int_0^\omega [p_{i1}(t, 0)\beta_1 + \cdots + p_{im}(t, 0)\cdot\beta_m]dt - \alpha\mu\beta_i + \mu^2(\cdots) \\ \frac{\partial(W_1, \ldots, W_m)}{\partial(\beta_1, \ldots, \beta_m)} = \mu^m \mid b_{ij} - \delta_{ij}\alpha \mid + \mu^{m+1}\Phi(\mu, \alpha) = 0 \end{gathered} \tag{14}$$

where $\Phi(\mu, \alpha)$ is a completely determined analytic function of μ and α. It follows from the particular form of the expansion (14), that if α_1 is a simple root of equation (5), then (on the basis of a theorem on implicit functions) the equation (13) will admit an analytic solution $\alpha_i(\mu)$, which reduces to α_1 when $\mu = 0$. It follows from this, that to every simple root of equation (5), there corresponds an analytic characteristic exponent. The solutions of (4), which correspond to these characteristic exponents, will also be analytic, for we obtain the functions $x_s(0) + \mu\, x_s(1) + \ldots$ from the periodic solutions of the auxiliary system by replacing β_i by the solution of the homogeneous system (12), which can always be chosen to be analytic in μ. This completes the proof of the theorem.

The following generalized theorem is also true.

Theorem II. We suppose that

$$P_i^{(\sigma)}(c_1,\ldots, c_m) \equiv 0 \quad (\sigma = 1,\ldots, k-1)$$

where

$$P_i^{(\sigma)} = \left[\frac{d^{\sigma-1}}{d\mu^{\sigma-1}}(W_i(c_1,\ldots,c_m,\mu,0))\right]_{\mu=0}$$

If $a = a^*$ is a simple root of the equation

$$\left|\frac{\partial P_i^{(k)}}{\partial c_j} - \delta_{ij}(\alpha)\right| = 0 \tag{15}$$

then the corresponding characteristic exponent will be an analytic function of the parameter μ of the form

$$\lambda = \lambda_1 + \mu^k\alpha^* + \mu^{k+1}\alpha_{k+1} + \cdots$$

In the case when $a = a^*$ is a multiple root of equation (5), then the corresponding characteristic exponent is, generally speaking, an analytic function of the parameter $\mu^{1/r}$ (where r is a positive integer less than the multiplicity of the root a^*) of the form

$$\lambda_1 + \mu\alpha^* + \mu^{\left(1+\frac{1}{r}\right)}\alpha_2 + \cdots$$

The determination of criteria for the analyticity of the characteristic exponents is, obviously, of interest in this case also.

Remark. If Re $\lambda_1 = 0$, the sign of Re (μa^*) determines, for small enough values of $|\mu|$, the sign of the real parts of the characteristic exponents corresponding to a^*.

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Translated by H.P.T.